The Binomial Model

The value of a derivative at t_0 is the present value of its expected value at maturity, that is:

$$f_0 = [pf_u + (1-p)f_d]e^{-rT} \qquad \textbf{eq. (1)}$$

Proof

Assuming that the underlying stock price is S_0 at the inception of the option, and that it can go up (S_0^u) or down (S_0^d) according to a factor that multiplies its original value:

$$S_0^u = S_0 u \qquad ; \qquad S_0^d = S_0 d$$

Assuming that we short a derivative and take a symmetric position on a Δ quantity of the underlying stock, we have a portfolio whose value at inception is:

$$\Pi_0 = S_0\Delta - f_0 \qquad \textbf{eq. (2)}$$

At the maturity of the option the underlying stock will go up or down, and, accordingly, the portfolio value will be: Π_T^u or Π_T^d.

$$\Pi_T^u = S_0 u\Delta - f_u \qquad ; \qquad \Pi_T^d = S_0 d\Delta - f_d$$

If we set $\Pi_T^u = \Pi_T^d$, the portfolio will be riskless, since, at time *T*, the portfolio will value the same in both the up and down scenarios.

This is possible if we find the appropriate Δ that equals the value of both portfolios under the two different situations (S_0^u and S_0^d). The required Δ is equal to:

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \qquad \textbf{eq. (3)}$$

In such a case, the appropriate discount rate will be the risk free rate. Assuming that the continuously compounded interest rate is *r*, then the current portfolio value will be either

$$\Pi_0 = [S_0 u\Delta - f_u]e^{-rT} \qquad \text{or} \qquad \Pi_0 = S_0\Delta - f_0$$

Therefore:

$$[S_0 u\Delta - f_u]e^{-rT} = S_0\Delta - f_0$$

From which we deduct:

$$f_0 = S_0\Delta - [S_0 u\Delta - f_u]e^{-rT}$$

If we multiply $S_0 u\Delta$ by $e^{rT}e^{-rT}$ and place e^{-rT} in evidence, we will arrive at:

$$f_0 = S_0\Delta e^{rT}e^{-rT} - [S_0 u\Delta - f_u]e^{-rT}$$

$$f_0 = [S_0\Delta e^{rT} - S_0 u\Delta + f_u]e^{-rT} \qquad \textbf{eq. (4)}$$

Replacing Δ [eq. (3)] in equation (4), we will get:

$$f_0 = \left[S_0 \frac{f_u - f_d}{S_0 u - S_0 d} e^{rT} - S_0 u \frac{f_u - f_d}{S_0 u - S_0 d} + f_u\right] e^{-rT}$$

We can now remove S_0:

$$f_0 = \left[S_0 \frac{f_u - f_d}{S_0 (u-d)} e^{rT} - S_0 u \frac{f_u - f_d}{S_0 (u-d)} + f_u\right] e^{-rT}$$

$$f_0 = \left[\frac{f_u - f_d}{u-d} e^{rT} - u \frac{f_u - f_d}{u-d} + f_u\right] e^{-rT}$$

We may now expand the equation:

$$f_0 = \left[\frac{f_u}{u-d} e^{rT} - \frac{f_d}{u-d} e^{rT} - u \frac{f_u}{u-d} + u \frac{f_d}{u-d} + f_u\right] e^{-rT}$$

Rearranging the equation in terms of the variables f_u and f_d:

$$f_0 = \left[\frac{f_u}{u-d} e^{rT} - u \frac{f_u}{u-d} + f_u + u \frac{f_d}{u-d} - \frac{f_d}{u-d} e^{rT}\right] e^{-rT}$$

In addition, placing f_u and f_d into evidence:

$$f_0 = \left[\left(\frac{e^{rT}}{u-d} - \frac{u}{u-d} + 1\right) f_u + \left(\frac{u}{u-d} - \frac{e^{rT}}{u-d}\right) f_d\right] e^{-rT}$$

$$f_0 = \left[\left(\frac{e^{rT}}{u-d} - \frac{u}{u-d} + \frac{u-d}{u-d}\right) f_u + \left(\frac{u}{u-d} - \frac{e^{rT}}{u-d}\right) f_d\right] e^{-rT}$$

$$f_0 = \left[\left(\frac{e^{rT} - u + u - d}{u-d}\right) f_u + \left(\frac{u - e^{rT}}{u-d}\right) f_d\right] e^{-rT}$$

$$f_0 = \left[\left(\frac{e^{rT} - d}{u-d}\right) f_u + \left(\frac{u - e^{rT}}{u-d}\right) f_d\right] e^{-rT} \qquad \textbf{eq. (5)}$$

Assuming:

$p = \frac{e^{rT} - d}{u-d}$ and $1 - p = \frac{u - e^{rT}}{u-d}$

Please note that:

$$1 - p = \frac{u-d}{u-d} - \frac{e^{rT} - d}{u-d} = \frac{u - d - e^{rT} + d}{u-d} = \frac{u - e^{rT}}{u-d}$$

We may now replace in equation (5):

$$f_0 = [p f_u + (1-p) f_d] e^{-rT}$$